SYNTHESIS BY ARCS ON THE UNIT CIRCLE

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ABSTRACT

Take a set of \( n \) arcs on the unit circle and consider the characteristic function of the set of \( n \) arcs. Think of this as a waveform, taking only the values 0 and 1. Produce a sound by generating a wavetable by sample and hold. You can think of this procedure as a very simple synthesizer. The question is: which sounds can be generated in this way. The surprising answer is: all of them. Thanks to a flourishing of very interesting papers from the Twenties to the early Fifties of last century, by Szegő, Verblunsky, Szasz, Friedman, Geronimus and Ghizzetti, among others, one can prove that it is possible to define sets of \( n \) arcs, whose characteristic functions have preassigned Fourier coefficients \( c_0, \ldots, c_{n-1} \).

In this paper we introduce such a synthesis procedure, give a sketch of the theorem, on which the procedure rests and highlight differences and similarities with Fourier synthesis and Shannon’s sampling theorem. We also discuss the mechanical principles on which a device performing this new kind of synthesis rests.

1. INTRODUCTION

Let \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \) be the unit circle in the complex plane. As customary, we think of \( S^1 \) as the quotient \( \mathbb{R}/2\pi\mathbb{Z} \), or, equivalently, as the unique compact manifold whose universal cover is \( \mathbb{R} \), the covering map being obviously \( \mathbb{R} \to \mathbb{C} \) and the map is bijective. Therefore we think of a function \( f : S^1 \to \mathbb{C} \) as a \( 2\pi \)-periodic function \( f : \mathbb{R} \to \mathbb{C} \), and to think of the integral of a summable function \( f \) on \( S^1 \) i.e. measurable with respect to \( \mu(S^1) = 1 \). Lebesgue measure on \( S^1 \), \( \int_{S^1} fd\mu = \frac{1}{2\pi} \int_0^{2\pi} f(t)dt \). (By abus de langage we will keep calling by the same name \( f \) both the function on the universal cover \( \mathbb{R} \) and the function on the covered manifold \( S^1 \).) From a mathematical point of view, a periodic waveform is a summable function \( f \) on \( S^1 \). Assume also that the amplitude is finite, so that \( f \) is bounded. After rescaling, you can of course assume that \( 0 \leq f(t) \leq 1 \). Let \( \mathbb{I} \) be the unit interval [0,1] and let \( L^1(S^1, \mathbb{I}) \) be the set of summable functions with values in \( \mathbb{I} \). Remark that \( L^1(S^1, \mathbb{I}) \) is not a vector space: all you can use from a functional analysis point of view is the structure of a metric space inherited from that of \( L^1(S^2, \mathbb{R}) \). In this sense, the theory which we are going to discuss is essentially nonlinear. To every \( f \in L^1(S^1, \mathbb{I}) \) associate the sequence \( \{ c_k(f) \}_{k \in \mathbb{Z}} \) of its Fourier coefficients defined by

\[
c_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{ikt}dt
\]

Remark that, being \( f \) \( \mathbb{R} \)-valued, one has \( c_{-k} = \overline{c_k} \). Remark also that we adopted the convention of calling \( c_k \) what is more frequently indicated with \( \tilde{c}_k \). The choice is of course, conceptually irrelevant.

Consider now the very special functions in \( L^1(S^1, \mathbb{I}) \) which are the characteristic functions of a set of \( n \) arcs on the unit circle. To fix the notations, let \( \xi_r, \eta_r, \ (r = 1, \ldots, n) \) be \( 2n \) real numbers such that \( 0 = \xi_1 < \eta_1 < \cdots < \xi_n < \eta_n < 2\pi \) and associate to them \( n \) arcs on the unit circle whose endpoints are, respectively, \( e^{i\xi_r} \) and \( e^{i\eta_r} \), \( r = 1, \ldots, n \). Let \( \chi \) be the characteristic function of the set of arcs, defined (on the universal cover \([0,2\pi)\)) by

\[
\chi(t) = \begin{cases} 
1 & \text{if } t \in \bigcup_{r=1}^{n}[\xi_r, \eta_r] \\
0 & \text{otherwise}
\end{cases}
\]

From now on we will refer to a characteristic function of \( n \) arcs, as a \( n \)-rectangular wave. We are now in a position to state the following

**Theorem 1** If \( f \in L^1(S^1, \mathbb{I}) \), then there exist a sequence \( \{ \chi_n \}_{n \in \mathbb{N}} \) of \( n \)-rectangular waves such that

\[
c_k(f) = c_k(\chi_n), \quad k = 0, \ldots, n-1.
\]

The sequence \( \{ \chi_n \}_{n \in \mathbb{N}} \) is not unique. Actually each \( \chi_n \) can be chosen freely in a 1-parameter family \( \{ \chi_{n,\lambda} \} \), where \( \lambda \in [0,2\pi) \).

**Theorem 2** Each of the sequences \( \{ \chi_{n,\lambda} \}_{n \in \mathbb{N}, \lambda \in [0,2\pi)} \), whose existence is assured by Theorem 1, weak * converges to \( f \) in \( L^1 \).

The consequences for sound synthesis are rather surprising: you can approximate any spectrum up to order \( n \) with a spectrum of a very simple rectangular waveform, taking only the values 0 and 1. It is rather easy to implement a software able to associate to any set of \( n \) arcs the sound produced by the characteristic function. The situation is somewhat similar to that of FM synthesis: one has a machine potentially capable of producing rich spectra by means of a very simple device. The problem is to tame the monster, that is to say, to find procedures to orient the timbre in the desired directions, by manipulating the parameters. Said that, there is a big difference with respect to FM synthesis: in this case, in principle, one is able to synthesize every possible sound, while, as far as we know, no characterization of FM sounds is available. We refer to [1] for a discussion of some aspects of the intricacies of mathematical FM theory.

In the next section we will indicate the main steps of a proof of Theorem 1. The exposition is very sketchy and uneven: we indulge in some easy, but, to our advice illuminating, computations and completely bypass some difficult steps. The intention is to convey the flavor of the mathematical tools employed and to highlight the noticeable fact that the proof is actually constructive.
A totally different proof of Theorem 1 can be given, following the guideline of Friedman (4), see also (3), which uses differential topology, instead of complex analysis, as the main tool. The differential-topological proof is maybe simpler and has the advantage to allow a remarkable generalization to compact manifolds other than $S^1$. The big drawback, as far as sound synthesis is concerned, is that it is not constructive and does not, at least for the moment, lead to implementable algorithms.

We do not discuss the proof of Theorem 2. It is just a brilliant exercise in pure mathematical analysis and does not seem (once more, at least for the moment) to shed any light on the synthesis algorithm. For the interested reader, we refer to the original proofs of Verblunsky [4] and Ghizzetti [5].

2. SKETCH OF THE PROOF OF THEOREM 1

We outline the main steps of the proof of the theorem. For the (many) missing details, we refer to the original papers of Szegő (6), Verblunsky [4], Szasz [3], Friedman [4], Geronimus [2] and Ghizzetti [5], or to the (somewhat more friendly) rational reconstruction contained in the PhD thesis of G. Caterina [10] (mainly concerned with the, also rather surprising, implications of the theorem for fuzzy logic.)

2.1. Fourier coefficients of $n$-rectangular waves

Let us start by computing the Fourier coefficients of a $n$-rectangular wave $\chi$ defined by the arcs $[e^{i\xi r}, e^{i\eta r}] \subset S^1$, $(r = 1, \ldots, n)$:

\[
\begin{align*}
\sigma_0(\chi) &= \frac{1}{2\pi} \int_0^{2\pi} \chi(t) dt = \frac{1}{2\pi} \sum_{r=1}^{n} (\eta_r - \xi_r), \\
\sigma_k(\chi) &= \frac{1}{2\pi} \int_0^{2\pi} \chi(t) e^{ikt} dt = \frac{1}{2\pi ki} \sum_{r=1}^{n} (e^{ik\eta_r} - e^{ik\xi_r}), \quad k \in \mathbb{Z}^*.
\end{align*}
\]

To prove the theorem means, given $n$ numbers $c_0, \ldots, c_{n-1}$ (which are the Fourier coefficients of a $f \in L^1(S^1, \mathbb{I})$). For a discussion of the necessary conditions see the references above), to find $2n - 1$ numbers $0 = \xi_1 < \eta_1 < \xi_2 < \eta_2 \cdots < \xi_n < \eta_n < 2\pi$, such that

\[
\begin{align*}
\sum_{r=1}^{n}(\eta_r - \xi_r) &= 2\pi c_0, \\
\sum_{r=1}^{n}(e^{i\eta_r} - e^{i\xi_r}) &= 2\pi i c_1, \\
&\vdots \\
\sum_{r=1}^{n}(e^{i(n-1)\eta_r} - e^{i(n-1)\xi_r}) &= 2\pi i c_{n-1}.
\end{align*}
\]

We are not aware of any simple way of solving these $n$ nonlinear equations in $2n - 1$ unknown. The solution which we will describe requires a rather complicated machinery.

2.2. A nonlinear Fourier transform

To every $f \in L^1(S^1, \mathbb{I})$, we associate a function $h_f$ holomorphic in the unit circle $D = \{z \in \mathbb{C} : |z| < 1\}$, defined by $h_f(z) = \exp(-2\pi i \sum_{k=1}^{\infty} c_k (f) z^k)$. Consider the power series expansion $h_f = \sum_{k=0}^{\infty} s_k z^k$ and define a sequence $s : \mathbb{Z} \rightarrow \mathbb{C}$ by $s(k) = s_k$, $k \geq 0$, $s(0) = 2 \sin \pi c_0$ and $s(-k) = \bar{s}_k$. From now on we will refer to $s_k$ ($k \in \mathbb{Z}$) as the $k$-th nonlinear coefficient of $f$.

Taking the logarithmic derivative of both sides of the identity

\[
e^{2\pi i \sum_{k=1}^{\infty} c_k z^k} = \sum_{k=1}^{\infty} s_k z^k
\]

one obtains

\[
-2\pi i \left(\sum_{k=1}^{\infty} k c_k z^{k-1}\right) \left(\sum_{k=1}^{\infty} s_k z^k\right) = \sum_{k=1}^{\infty} k s_k z^{k-1}.
\]

From the last identity one sees that the nonlinear coefficients are given as polynomials in the Fourier coefficients, and, reciprocally, that the Fourier coefficients are given as polynomials in the nonlinear coefficients:

\[
s_k = P(c_1, \ldots, c_{n-1}), \quad c_k = Q(s_1, \ldots, s_{k-1})
\]

Let us call $N$ the map which associates to a function $f$ its nonlinear coefficients $s_k$. One could think of this map as a nonlinear Fourier transform and of the set of nonlinear coefficients as a nonlinear spectrum. The abstract scheme is illustrated by the following commutative diagram

\[
L^1(S^1, \mathbb{I}) \xrightarrow{C_H^\mathbb{I}} \mathbb{C}^*_h \xrightarrow{c_k \mapsto c_k} \mathbb{C}^*_h
\]

where $C_H^\mathbb{I}$ is the space of all the hermitian sequences, the vertical arrow is the Fourier transform, the diagonal arrow the nonlinear Fourier transform $N$ and the horizontal arrow the algebraic map between Fourier and nonlinear coefficients, given by the polynomial $P$.

2.3. The holomorphic function associated to a $n$-rectangular wave

A first hint on the usefulness of the procedure described above comes from an explicit calculation of the holomorphic function associated to a $n$-rectangular wave. Let us, as usual, denote by $\chi$ the rectangular wave defined by the arcs $[e^{i\xi r}, e^{i\eta r}] \subset S^1$, $(r =
1, . . . , n). Then the associated holomorphic function \( h_k \) is given by
\[
\sum_{k=1}^{\infty} c_k(\chi) z^k = \sum_{k=1}^{\infty} \left( -\frac{1}{2\pi i} \sum_{r=1}^{n} (-e^{ik\eta r} + e^{ik\xi r}) z^k \right).
\]
Recalling that, for \( s \),
\[
\text{Proposition 1} \quad \text{The } n \text{-rectangular wave associated to the nonlinear coefficients }
\]
\[
\text{Proposition 2} \quad \text{The } n \text{-rectangular wave associated to the collection of } n \text{ Toeplitz matrices verifying the conditions of Proposition 1, is the one defined by the arcs } [e^{ik\xi}, e^{in\eta}], \text{ whose left endpoints } e^{ik\xi} \text{ and right endpoints } e^{in\eta} \text{ are, respectively, the roots of the following two polynomials of degree } n:
\]
\[
\frac{1}{\pi} \sum_{r=1}^{n} \left( \ln \left( -e^{ik\eta r} + 1 \right) - \ln \left( -e^{ik\xi r} + 1 \right) \right) = 0,
\]
Exponentiating allows to get rid of the unpleasant logarithm in the last formula. To sum up, for the holomorphic function associated to the rectangular wave \( \chi \) one computes the expression
\[
\exp \left( \sum_{r=1}^{n} \ln \left( 1 - \frac{e^{ik\eta r}}{e^{ik\xi r}} \right) \right) = \prod_{r=1}^{n} \exp \left( \ln \left( 1 - \frac{e^{ik\eta r}}{e^{ik\xi r}} \right) \right) = \prod_{r=1}^{n} \frac{1 - e^{ik\eta r}}{1 - e^{ik\xi r}}
\]
which is indeed very nice. Observe that the function, holomorphic on the (open) disk, has poles at the left endpoints and zeroes at the right endpoints of the arcs.

### 2.4. Toeplitz matrices of nonlinear coefficients

Associate to the nonlinear coefficients \( s_k \), \( k \in \mathbb{Z} \) of a \( f \in L^1(S^1, \mathbb{I}) \) a sequence \( \{ T_k \} \) of \( (k+1 \times k+1) \) Toeplitz matrices (i.e. constant on the NW to SE diagonals), by
\[
T_k = \begin{pmatrix}
s_0 & s_1 & s_2 & \ldots & s_k \\
-s_1 & s_0 & s_1 & \ldots & s_{k-1} \\
& \ldots & \ldots & \ldots & \ldots \\
-s_{k-1} & s_{-k} & s_{-k+1} & \ldots & s_0
\end{pmatrix}
\]
Observe that \( T_k \) is hermitian and therefore \( D_k = \det T_k \) is a real number. One can use the Toeplitz determinants \( D_k \) to characterize the \( n \)-rectangular waves, according to the following

\[
\text{Proposition 1} \quad f \in L^1(S^1, \mathbb{I}) \text{ coincides with a } n \text{-rectangular wave almost everywhere if and only if } D_k > 0 \text{ for } k = 0, 1, \ldots, n-1 \text{ and } D_k = 0, \forall k \geq n. \text{ If this is not the case, then } D_k > 0, \forall k \in \mathbb{N}.
\]

The above proposition establishes a bijection between \( n \)-rectangular waves and collections of \( n \) Toeplitz matrices verifying the above conditions. But one can prove more. An explicit description of the bijection is given by the following

\[
\text{Proposition 2} \quad \text{The } n \text{-rectangular wave associated to the collection } T_k, \( k = 1, \ldots, n \) \text{ of Toeplitz matrices verifying the conditions of Proposition 1, is the one defined by the arcs } [e^{ik\xi}, e^{in\eta}], \text{ whose left endpoints } e^{ik\xi} \text{ and right endpoints } e^{in\eta} \text{ are, respectively, the roots of the following two polynomials of degree } n:
\]

\[
\frac{1}{\pi} \sum_{r=1}^{n} \left( \ln \left( -e^{ik\eta r} + 1 \right) - \ln \left( -e^{ik\xi r} + 1 \right) \right) = 0,
\]

Exponentiating allows to get rid of the unpleasant logarithm in the last formula. To sum up, for the holomorphic function associated to the rectangular wave \( \chi \) one computes the expression
\[
\exp \left( \sum_{r=1}^{n} \ln \left( 1 - \frac{e^{ik\eta r}}{e^{ik\xi r}} \right) \right) = \prod_{r=1}^{n} \exp \left( \ln \left( 1 - \frac{e^{ik\eta r}}{e^{ik\xi r}} \right) \right) = \prod_{r=1}^{n} \frac{1 - e^{ik\eta r}}{1 - e^{ik\xi r}}
\]
which is indeed very nice. Observe that the function, holomorphic on the (open) disk, has poles at the left endpoints and zeroes at the right endpoints of the arcs.
From the lemma, one immediately has that \( \forall \lambda \in [0, 2\pi) \) there is one and only one solution to the following equation in the unknown \( s \):

\[
| \begin{array}{cccccc}
 s_1 & s_2 & s_3 & \ldots & s \\
 s_0 & s_1 & s_2 & \ldots & s_{n-1} \\
 \ldots & \ldots & \ldots & \ldots & \ldots \\
 s_{-n+2} & s_{-n+3} & s_{-n+4} & \ldots & s_1 \\
 \end{array} | = 0
\]

Putting in the matrix \( T_n \), \( s_n \) equal to the solution of the equation above, and using the well-known (in the Toeplitz world) fact that \( D_{n-1}^2 - D_{n-1}D_n = |F_n| \), one concludes that \( D_n = 0 \), and the theorem is proved.

Figure 2: Rectangular waveform associated to the set of arcs shown in fig. 1.

3. DIFFERENCES AND SIMILARITIES WITH FOURIER SYNTHESIS

In Fourier analysis one has a set of elementary signals, the trigonometric functions, by means of which one can reconstruct every periodic signal. In our case, the elementary signals are the characteristic functions of sets of \( n \) arcs on the unit circle, by means of which one can reconstruct every bounded periodic signal. Reconstruction in Fourier world means that a signal is an infinite linear combination of elementary signals. In our case the reconstruction is completely different. We still have a sequence of elementary signals, completely determining the original signal, but, instead of obtaining the original signal as the sum of a series, one obtains it by a limit process: the original signal is the weak * limit in \( L^1 \) of the elementary signals.

For what concerns approximation, which is the relevant aspect for sound synthesis, an actual signal is spectrally approximated up to order \( n \) by a trigonometric polynomial of order \( n \) in Fourier synthesis, while it is approximated by a \( n \)-rectangular wave in our case. The obvious difference being that, while a rectangular wave is a much simpler function, both algebraic and computational points of view, a trigonometric polynomial is band-limited, while a rectangular wave is not. Therefore, while in Fourier case one has a simple estimate of the approximation error: the tail of the series of the original signal, in our case the error is given by a much more complicated object: the difference of the tails of the original series and the series of the \( n \)-rectangular wave.

This brings in a third fundamental difference between ours and the Fourier scheme: while the approximating trigonometric polynomial of order \( n \) is unique, in our case, according to the theorem, one has a choice of the approximating \( n \)-rectangular wave, depending on a parameter \( \lambda \in [0, 2\pi) \). The wide-open question is how to use this freedom in the choice of the approximating rectangular waves to the scope of minimizing the error. The problem seems to be a difficult theoretical one. Some first results are contained in [11].

Experimenting with sounds generated by \( n \)-rectangular waves is currently carried over at the sound laboratory Musica Inaudita at the University of Salerno, by means of a synthesizer, called Arcosint[1] whose mechanical principles are discussed in section 5.

4. FORMAL ANALOGIES WITH SHANNON’S SAMPLING THEOREM

There is an interesting formal analogy between Theorems 1 and 2, and Shannon’s sampling theorem that seems to deserve some comments. While the concrete framework is totally different (in Shannon’s case one deals with bandlimited signals on the real line, in ours with, so to say, stripvalued periodic signals) there is a common abstract framework. In both cases you restrict to a subset of signals and you end up with a reconstruction theorem in terms of a privileged class of signals (sinc functions and rectangular waves, respectively.)

The reconstruction is, of course, totally different: a bandlimited signal as the sum of a series of sinc functions, having as coefficients samplings of the signal, the stripvalued signal as weak * limit of a sequence of rectangular waves.

Yet the philosophy is similar: selecting a special class of signals which can be totally described in terms of a very special elementary subclass.

5. \( n \)-RECTANGULAR WAVE GENERATION: A MECHANICAL INSIGHT

The theorem stated in this paper suggests the possibility to approximate a given spectrum with a rectangular wave obtained as a characteristic function of a set of arcs on the unit circle.

From a purely mechanical point of view the problem of a good approximation (up to the \( n \)-th term) of the Fourier spectrum, is turned to the problem of the production of such a mechanical wave. The more flexible (customizable) and practical way to build a rectangular wave is by using the diaphragm of a loudspeaker, moving between two fixed endpoints. Think of these two endpoints as the rest position of the membrane and its position at a fixed mechanical tension. The rest position corresponds to a zero driving current (the loudspeaker electromagnet, when the driving current is zero, is inactive so the membrane is naturally relaxed). The fixed position corresponds to a fixed-value driving current, for example the maximum elongation (i.e. the maximum strength of the diaphragm) corresponds to the maximum current with which it is possible to feed the device. Using a loudspeaker as a transducer electric current \( \rightarrow \) position of a mechanical membrane, we changed the problem to the one of producing a current which changes between two values. The changes in time of the current between the two endpoints will drive automatically switches of the membrane between two positions at the same rate, producing a rectangular wave. The limits on the frequencies physically obtainable with this method are of course the ones of the response in time of the membrane.

For example, to build a square wave whose period is 10 ms (frequency \( f = 1/0.01 = 100 \) Hz), duty cycle 50% (i.e. the wave assumes the value 1 for one half of the period and 0 for the

[1]The name intends to be a homage to the architect Paolo Soleri, the visionary builder of the utopian Renaissance-like city of Arcosanti in the Arizona desert.
second half), we have to produce a current which has exactly the same properties, alternating between zero and a fixed value with the same duty cycle and in the same time intervals. Maybe the best (surely one of the most customizable and cost effective) way to generate such a current, able to correctly drive loudspeakers, is by employing a computer-controlled sound card. The personal computer, through a software description of the waveform will instruct the sound card to produce the desired current. Sound cards take as input a stream of digital values (representing the waveform) from a program and through an interpolation process (typically linear fit between adjacent values, or sample and hold) it produces the driving current. Suppose we are working with a 44.1 KHz sound card, and suppose that we want to build the 50% duty cycle rectangular wave, we were talking about above. 44.1 KHz means that one second of signal is described by a series of 44100 values. To tell the sound card to produce the right switching current we have to provide a sequences of 22050 ones followed by a sequence of 22050 zeroes.

Summarizing the whole process to produce a mechanical rectangular wave, we start from a program (the virtual synthesizer) able to produce the right series of values to represent (taking into account the work frequency of the sound card) the rectangular wave; the sound card will translate this series (through a sample-and-hold procedure) into a current with exactly the same functional profile; finally the loudspeaker will turn this current into a mechanical wave.

It could be useful to spend a few more words about the sound card working frequency. If \( f_{sc} \) (sc stands for sound card) is such a frequency, the maximum number of variations per second we can describe is \( f_{sc}/2 \). For example, if \( f_{sc} = 44100 \), we cannot switch between 0 and 1 more than 22050 times per second (i.e., \( 1010101010 \ldots \)). The upper limit for the switching frequency has a direct influence on the accuracy of the spectral approximation capabilities of the synthesized waveform. The Fourier spectrum of the rectangular wave generated following the procedure described in the proof of the theorem, has exactly the first \( n \) spectral coefficients identical to the ones of the function \( f \in L^1(S^1, 1) \). Each arc is identified by its endpoints, so to each arc are associated two flips in the characteristic function diagram, the first one from 0 to 1, and the second one from 1 to 0. If the working frequency of the sound card is \( f_{sc} \), the shortest possible arc describable is one of length \( 1/f_{sc} \). Therefore we cannot have more than \( f_{sc}/2 \) arcs of length \( 1/f_{sc} \), spaced by minimal intervals of length \( 1/f_{sc} \). If the sound card frequency is 44100 Hz, this means that the maximum number of arcs is 22050. This is also the upper limit to the number of Fourier coefficients that the rectangular waveform will have in common with the function. The spectral accuracy limit is set to \( f_{sc}/2 \) number of partials, which allows for a more than satisfactory synthesis procedure, even on a typical PC sound card.

## 6. Conclusions

We proposed a new sound synthesis method based on a rather complicated pure-mathematical machinery. Still, we think that the outcome is worth the price: a virtual machine able to synthesize any sound with rectangular waves, the simplest possible functions from a logical point of view (boolean functions). We discussed differences and similarities with additive synthesis and underlined some possible draw-backs of our method. A lot more of experimental and theoretical work is required. The good news, from a pure-mathematical point of view, are that the framework in which the theorem we stated stands, the Szegő theory of trigonometric polynomials on the unit circle, longtime considered as a beautiful but sort of a marginal subject, is now back in the mainstream of mathematical research. This opens up the possibility, for people interested in sound synthesis, to go fishing for hints in the hundreds of papers published in the last few years on the subject (see the recent authoritative treatise of Barry Simon [12, 13], in two volumes and more than thousand pages.) On the experimental side, a software implementation in Java is under development and will be available at [http://www.musicainaudita.it/arco](http://www.musicainaudita.it/arco). At the same URL, one will be also able to find the sonic results of ongoing playing-around with distributions of arcs on the unit circle. Along with the virtual synthesizer, a hardware implementation with discrete components (2\(n\) monostables) is currently under study, in collaboration with A. De Nardo and G. Lisi of the department of Electronic Engineering (DIIIE) of the University of Salerno.

## 7. References


[12] B. Simon, Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, ser. AMS Colloquium Publica-