Robust Design of Very High-Order Allpass Dispersion Filters

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Dispersive Propagation in a Spring Reverb

- The torsional mode typically used by spring reverberators is highly dispersive, giving the spring its characteristic sound.
Dispersive Waveguide

- Each spring element is modeled using a set of dispersive waveguide sections.
- Left-going and right-going waves are separately processed via delay elements and commuted dispersion and loss filters.
Allpass Dispersion Filter Design

\[ G(z) = \frac{\rho_N + \rho_{N-1}z^{-1} + \cdots + \rho_1z^{-N+1} + z^{-N}}{1 + \rho_1z^{-1} + \cdots + \rho_{N-1}z^{-N+1} + \rho_Nz^{-N}} \]

- Hilbert transform methods
  - Yegnanarayana, IEEE-ASSP 1982
  - Reddy and Swamy, ICASSP 1998
  - Filter phase (not group delay) matched
  - Potential time aliasing, numerical issues; not in factored form

- Optimal filter design formulation
  - Lang and Laakso, IEEE-CAS1 1994; Lang, IEEE-SP 1998
  - Rocchesso and Scalcon, IEEE-CAS 1996
  - Bensa et al., ASA 2004
  - Rauhala and Valamaki, IEEE-SPL 2006
  - Maximum order limited by numerical difficulties; expensive design
First-Order Allpass Filter

\[ G(z) = \frac{-\rho e^{-j\theta} + z^{-1}}{1 - \rho e^{-j\theta} \cdot z^{-1}} \]

\[ \tau(\omega) = \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\omega - \theta)} \]

**transfer function**

**group delay**
First-Order Allpass Group Delay

- As the pole is moved toward the unit circle, the first-order allpass group delay $\tau(\omega)$ becomes more peaked about the pole angle $\theta$ – the maximum increases, and the peak narrows.
First-Order Allpass Group Delay

- The integral of the group delay $\tau(\omega)$ of a first-order allpass filter is $2\pi$, independent of $\rho$ and $\theta$,

$$
\int_{0}^{2\pi} \tau(\omega) d\omega = \varphi(2\pi) - \varphi(0) = 2\pi.
$$
Allpass Filter Design Approach

- Integrate \( \tau(\omega) \), and add a constant delay \( \tau_0 \) such that \( \pi(\omega) + \tau_0 \) integrates to a multiple of \( 2\pi \).
- Divide \( \tau(\omega) + \tau_0 \) into \( 2\pi \)-area frequency bands.
- Fit a first-order allpass filter section to each band.
First-Order Allpass Design

- The pole angle \( \theta \) is the band midpoint,
  \[ \theta = (\omega_- + \omega_+) / 2 \]

- The section pole radius \( \rho \) is chosen to make the band edge group delay a fraction \( \beta \) of its maximum.

\[ \rho = \eta - \left( \eta^2 - 1 \right)^{1/2} \]
\[ \eta = \frac{1 - \beta \cos \delta}{1 - \beta} \]
\[ \delta = (\omega_- - \omega_+) / 2 \]
Dispersion Filter Design Cost

- The design method is very inexpensive and may be used to update dispersion filters in real time.
  - The pole angles $\theta_k$ directly encode the dispersive delay $\tau(\omega)$, and may be efficiently computed.
  - The pole radii $\rho_k(\beta)$ control delay smoothing, and are roughly linear in section bandwidth.

\[ \tau(\omega) \]

\[ \rho \approx 1 - \left[ \frac{\beta}{1 - \beta} \right]^{1/2} \delta, \quad \delta \ll 1 \]

\[ \delta = \frac{1}{2} |\omega_+ - \omega_-| \]
Design Example: Spring Reverberator Element

- Poles, zeros follow smooth trajectories.
Adjusting $\beta$

Adjusting $\beta$ trades ripple for responsiveness to narrow-band group delay changes.
Increasing Model Order

- Adding a constant delay $\tau_0$ to the group delay $\tau(\omega)$ allows additional allpass sections to be used, and provides a more accurate fit.
Low-Frequency Modeling

- By setting $\tau(\omega) = 0$ outside the band of interest, the model order may be reduced.
Piano String Propagation Filter Design

\[ \exp\left\{ -\alpha(\omega) \cdot d - j\left[ \frac{\omega}{c_0} - \varphi(\omega) \right] \cdot d \right\} \]

 measured (-r) and modeled (-k) transfer function magnitude

 measured (-r) and modeled (-k) group delay, 64 biquads.

 measured (-r) and modeled (-k) impulse responses.

 transfer functions

 impulse responses
Stiff String Propagation Filter Design

\[ \tau(\omega) = \frac{\tau_0}{\sqrt{1 + B\omega^2}} \]

\[ \varphi(\omega) = \frac{\tau_0}{\sqrt{B}} \cdot \text{asinh}\sqrt{B\omega} \]
Summary

• New method for allpass dispersion-filter design:
  – Simple, numerically robust, nonparametric
  – Model order automatically determined
  – Filters produced in factored biquad form

• Future work
  – Applications
    • Strings, springs and tubes of all kinds
    • Filter group-delay equalization
  – Extensions
    • Multiband group-delay filter design
    • Time-varying group-delay design
    • Frequency-dependent smoothing parameter $\beta$